Carrollian Fluids and Spontaneous Breaking of Boost Symmetry

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In the hydrodynamic regime, field theories typically have their boost symmetry spontaneously broken due to the presence of a thermal rest frame although the associated Goldstone field does not acquire independent dynamics. We show that this is not the case for Carrollian field theories where the boost Goldstone field plays a central role. This allows us to give a first-principles derivation of the equilibrium currents and dissipative effects of Carrollian fluids. We also demonstrate that the limit of vanishing speed of light of relativistic fluids is a special case of this class of Carrollian fluids. Our results shine light on the thermodynamic properties and thermal partition functions of Carrollian field theories.

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Introduction.—In the past few years Carrollian physics, emerging by taking the limit of the vanishing speed of light, has been found useful for describing a variety of phenomena in contexts ranging from black holes [1–4], cosmology [5,6], gravity [3,7–16], to hydrodynamics [4,5,17–24]. Concretely, Carrollian fluids can be used to describe Bjorken flow, which is relevant for models of the quark-gluon plasma, cf. [23] (and its conformal generalization, Gubser flow [25]). Carrollian fluids also model dark energy in inflationary models [5]. Furthermore, Carrollian symmetries are expected to have a role to play in exotic phases of matter (e.g., via Carroll-fracton dualities [26–29] and in superconducting twisted bilayer graphene [30]).

Many of the properties encountered in this Carrollian limit are expected to be explained from underlying quantum field theories with inherent Carrollian symmetries. Indeed, if conformal symmetry is present in addition, such theories would be putative holographic duals to flat space gravity [31–41]. However, when attempting to formulate such Carrollian field theories, several issues have been pointed out including violations of causality, lack of well-defined thermodynamics, and ill-defined partition functions [5,24]. Our goal is to show that the lack of well-defined thermodynamics in Carrollian field theories is expected in the hydrodynamic regime, but that this issue can be cured when carefully accounting for the Carrollian symmetries.

The approach we take is to consider the hydrodynamic regime of such putative Carrollian field theories and show how to construct their equilibrium partition function and near-equilibrium dynamics. In particular, we will show that there is no proper notion of temperature in Carrollian fluids unless the Goldstone field of spontaneous broken boost symmetry is taken into account. This allows us to construct a well-defined hydrodynamic theory of Carrollian fluids (similar to *framids* in the language of [42,43]). In the process, we show that seemingly different approaches to Carrollian hydrodynamics previously pursued in the literature [5,17–24] are in fact equivalent and special cases of the Carrollian fluids we derive.

The fact that boost symmetry is spontaneously broken in hydrodynamics is not unexpected. Thermal states break the boost symmetry spontaneously due to the presence of a preferred rest frame aligned with the thermal vector [43,44]. In the context of hydrodynamics the thermal vector is the combination u^μ/T of the unit normalized fluid velocity u^μ and temperature T. On the other hand, the Goldstone field associated with the breaking of boost symmetry does not typically feature in the low energy spectrum of the theory because it is determined in terms of the other dynamical fields (see, e.g., [42]). This is easy to show for relativistic fluids. Consider a $(d+1)$-dimensional spacetime metric $g_{\mu\nu} = E^A_\mu E^\mu_A$ where E^A_μ is the set of vielbeins, $\mu = 0, \ldots, d$ are spacetime indices, and $A = 0, \ldots, d$ are internal Lorentz indices. The Goldstone field associated to spontaneous breaking of Lorentz boost symmetry is the vector $\ell_A = \Lambda_A^0$ where Λ_A^0 is the Lorentz boost matrix [42], and acquires an expectation value $\langle \ell_A \rangle = \delta_A^0$ in the ground state. In thermal equilibrium we can construct an equilibrium partition function $S = \int d^{d+1}x\sqrt{-g}P(T, u^\mu \ell_\mu)$ where P is the fluid pressure. The temperature is given by $T = T_0/|K|$ with T_0 a global constant temperature, and the fluid velocity is $u^\mu = K^\mu/|K|$ satisfying $u_\mu u^\mu = -1$ and defined in terms of the Killing (thermal) vector K^μ with modulus $|K|^2 = -g_{\mu\nu}K^\mu K^\nu$. We furthermore defined $\ell_\mu = E_\mu^A \ell_A$ satisfying $\ell_\mu \ell^\mu = -1$. The Goldstone equation of motion obtained from varying the partition function S with respect to ℓ_A is [42]

$$(\delta_\nu^\mu + \ell^\mu \ell_\nu)\frac{\delta S}{\delta \ell_\mu} = 0. \tag{1}$$

This equation implies $u^\mu = -\ell_\nu u^\nu \ell^\mu$ for arbitrary thermodynamic coefficients, and can only be satisfied if $\ell^\mu = u^\mu$. Indeed, we see that the dynamics of the Goldstone field ℓ^μ is determined by the dynamics of the fluid velocity and hence can be removed from the hydrodynamic description. The same conclusion is reached for the case of spontaneous breaking of Galilean (Bargmann) boost symmetry [45] (see also [44]). However, in the case of Carrollian symmetry, as we will show, the boost Goldstone acquires its own independent dynamics. Below we introduce Carrollian geometry and use it to show that, naively, there is no well-defined notion of temperature.

Carrollian geometry and the lack of temperature.—A *weak* Carrollian geometry on a $(d+1)$-dimensional manifold M is defined by a Carrollian structure $(v^\mu, h_{\mu\nu})$ consisting of the nowhere-vanishing Carrollian vector field v^μ, and the corank-1 symmetric tensor $h_{\mu\nu}$, the "ruler," satisfying $h_{\mu\nu}v^\nu = 0$. It is useful to define inverses $(\tau_\mu, h^{\mu\nu})$ satisfying $v^\mu \tau_\mu = -1$, and $\tau_\mu h^{\mu\nu} = 0$, as well as the completeness relation $-v^\mu \tau_\nu + h^{\mu\rho}h_{\rho\nu} = \delta_\nu^\mu$. It is also useful to introduce the spatial vielbeins e_μ^a and their inverses e_a^μ which can be used to write $h_{\mu\nu} = e_\mu^a e_{a\nu}$. Under Carrollian boosts, the inverses transform as

$$\delta_C \tau_\mu = \lambda_\mu, \quad \delta_C h^{\mu\nu} = 2\lambda_\rho h^{\rho(\mu}v^{\nu)}, \tag{2}$$

corresponding to $\delta_C e_a^\mu = v^\mu \lambda_a$, where we have used that $v^\mu \lambda_\mu = 0$. A *strong* Carrollian geometry is a weak Carrollian geometry together with an affine connection. We focus mainly on weak Carrollian geometry but discuss strong Carrollian geometry in Appendix D of the Supplemental Material [46].

Given the Carroll geometry and the existence of a thermal vector in equilibrium, namely the spacetime Killing vector k^μ, one can proceed as for other (non)-Lorentzian field theories [54–65] and construct an equilibrium partition function by identifying the invariant scalars under all local symmetries which the pressure P can depend on, as above Eq. (1) for the relativistic case. A more thorough construction of the partition function will be given in a later section. Here we note that for non-relativistic theories the temperature T is given by the scalar $T = T_0/(k^\mu \tau_\mu)$ with T_0 a constant global temperature. However, in the Carrollian case T is not invariant under boost transformations since $\delta_C(k^\mu \tau_\mu) = k^\mu \lambda_\mu \neq 0$. Indeed, there is no well-defined notion of temperature for arbitrary observers [66]. This is rooted in the fact that it is not possible to impose a timelike normalization condition on spacetime vectors such as the fluid velocity since $u^\mu \tau_\mu$ is not boost invariant. This argument does not rely on any specific model of Carrollian (quantum) field theory since these statements are valid in the hydrodynamic regime. This suggests that well-defined thermodynamic limits of such putative theories are subtle. Below we introduce the boost Goldstone and use it to show that it can be used to define an appropriate notion of temperature.

The Carroll boost Goldstone.—We define the boost Goldstone as the vector θ^μ which transforms under Carrollian boosts λ_μ as

$$\delta_C \theta^\mu = -h^{\mu\nu}\lambda_\nu, \tag{3}$$

where $\lambda_\mu v^\mu = 0$. This implies that only the spatial part of θ^μ is physical, which we can enforce by endowing the Goldstone with a timelike Stueckelberg symmetry of the form

$$\delta_S \theta^\mu = \chi v^\mu, \tag{4}$$

where χ is an arbitrary function [67]. With this we can build the boost and Stueckelberg invariant vielbeine

$$\hat{\tau}_\mu = \tau_\mu + h_{\mu\nu}\theta^\nu, \quad \hat{e}_a^\mu = e_a^\mu + v^\mu \theta^\mu e_{\mu a}, \tag{5}$$

which lead to the following invariant ruler:

$$\hat{h}^{\mu\nu} = \delta^{ab}\hat{e}_a^\mu \hat{e}_b^\nu = h^{\mu\nu} + v^\mu v^\nu(\theta^2 + 2\tau_\rho \theta^\rho) + 2v^{(\mu}\theta^{\nu)}, \tag{6}$$

where $\theta^2 = h_{\mu\nu}\theta^\mu \theta^\nu$. Together with v^μ and $h_{\mu\nu}$, these form an *Aristotelian* structure [68],

$$\hat{h}^{\mu\nu}\hat{\tau}_\nu = 0, \quad v^\mu \hat{\tau}_\mu = -1, \quad \hat{h}^{\mu\rho}h_{\rho\nu} = \delta_\nu^\mu + v^\mu \hat{\tau}_\nu =: \hat{h}_\nu^\mu, \tag{7}$$

that is partly dynamical due to the Goldstone θ^μ. The low-energy effective action for the Carrollian boost Goldstone is a two-derivative Hořava–Lifshitz type action as we show in Appendix A of [46]. If coupled to Carrollian gravity the resultant action would be derivable from the limit of the vanishing speed of light of the Einstein-Aether theory [69]. Before showing how the Goldstone allows one to define a notion of temperature, we first discuss the currents and conservation laws.

Currents and conservation laws.—We now consider an arbitrary fluid functional (or free energy) $S[\tau_\mu, h_{\mu\nu}; \theta^\nu]$ for a Carrollian fluid with spontaneously broken boosts. The variation of this functional is

$$\delta S = \int d^{d+1}xe\left[-T^\mu\delta\tau_\mu + \frac{1}{2}\mathcal{T}^{\mu\nu}\delta h_{\mu\nu} - K_\mu\delta\theta^\mu\right], \quad (8)$$

where T^μ is the energy current, $\mathcal{T}^{\mu\nu}$ the stress-momentum tensor, and K_μ the response to the Goldstone field. In particular $K_\mu = 0$ gives the analogous equation of motion for the Goldstone as in (1). The measure is defined as $e = \det(\tau_\mu, e^a_\mu) = \det(\hat{\tau}_\mu, e^a_\mu)$. The Ward identities for the Stueckelberg and boost symmetries are, respectively,

$$v^\mu K_\mu = 0, \quad T^\nu h_{\nu\mu} = K_\mu. \quad (9)$$

The equation of motion for the Goldstone, $K_\mu = 0$, imposes the condition $T^\nu h_{\nu\mu} = 0$. In other words, the boost Ward identity now becomes the equation of motion for the Goldstone. The momentum-stress tensor is not boost invariant. In fact, computing the second variation, which must vanish $\delta_C(\delta_C S) = 0$, we find that $\delta_C T^\mu = \delta_C K_\mu = 0$ and $\delta_C\mathcal{T}^{\mu\nu} = 2T^{(\mu}h^{\nu)\rho}\lambda_\rho$. The associated energy-momentum tensor (EMT) $T^\mu_\nu = -\tau_\nu T^\mu + \mathcal{T}^{\mu\rho}h_{\rho\nu}$ is also not boost invariant and transforms as

$$\delta_C T^\mu_\nu = K_\nu h^{\mu\rho}\lambda_\rho, \quad (10)$$

where we used (9). Doing the same for the Stueckelberg symmetry, the condition $\delta_S(\delta_S S) = 0$ implies that $\delta_S T^\mu_\nu = -\chi v^\mu K_\nu$. Hence, the EMT is both boost and Stueckelberg invariant if $K_\mu = 0$. The diffeomorphism Ward identity reads as

$$e^{-1}\partial_\mu(eT^\mu_\rho) + T^\mu\partial_\rho\tau_\mu - \frac{1}{2}\mathcal{T}^{\mu\nu}\partial_\rho h_{\mu\nu} = 0, \quad (11)$$

where we used that $K_\mu = 0$. It is possible to obtain manifestly boost invariant currents, including the EMT, by formulating the action in terms of the effective Aristotelian structure (5), as we show in Appendix B of [46].

Equilibrium partition function and Carrollian fluids.—To derive the currents of Carrollian fluids, we consider the equilibrium partition function construction. An equilibrium Carrollian background consists of a set of symmetry parameters $K = (k^\mu, \lambda^\mu_K, \chi_K)$, where k^μ is a Killing vector and λ^μ_K is a boost symmetry parameter, while χ_K is a Stueckelberg symmetry parameter. The various structures transform according to

$$\begin{aligned}
&\delta_K v^\mu = \pounds_k v^\mu = 0, \quad \delta_K\tau_\mu = \pounds_k\tau_\mu + \lambda^K_\mu = 0,\\
&\delta_K h_{\mu\nu} = \pounds_k h_{\mu\nu} = 0,\\
&\delta_K h^{\mu\nu} = \pounds_k h^{\mu\nu} + 2\lambda^K_\rho h^{\rho(\mu}v^{\nu)} = 0,\\
&\delta_K\theta^\mu = \pounds_k\theta^\mu - h^{\mu\nu}\lambda^K_\mu + \chi^K v^\mu = 0.
\end{aligned} \quad (12)$$

The boost and Stueckelberg symmetry parameters transform as

$$\delta\lambda^K_\mu = \pounds_\xi\lambda^K_\mu - \pounds_k\lambda_\mu, \quad \delta\chi^K = \pounds_\xi\chi^K - \pounds_k\chi, \quad (13)$$

under infinitesimal diffeomorphisms generated by ξ^μ, infinitesimal Carrollian boosts λ_μ and Stueckelberg transformations χ. As we show in Appendix B of [46], λ^K_μ and χ^K will not play a role in the effective fluid description.

Before enumerating the possible invariant scalars, we must provide a gradient ordering. As usual we take the geometry itself to be of ideal order, that is, $\tau_\mu \sim h_{\mu\nu} \sim v^\mu \sim h^{\mu\nu} \sim \mathcal{O}(1)$. Since θ^μ enters the definition of $\hat{\tau}_\mu$ it must have the same ordering, $\theta^\mu \sim \mathcal{O}(1)$. Gradients of these structures are $\mathcal{O}(\partial)$ and hence suppressed in a hydrodynamic expansion. Given this gradient scheme the only two ideal order invariants are

$$T = T_0/\hat{\tau}_\mu k^\mu, \qquad \vec{u}^2 = h_{\mu\nu}u^\mu u^\nu, \quad (14)$$

where $u^\mu = k^\mu/\hat{\tau}_\rho k^\rho$, which satisfies $\hat{\tau}_\mu u^\mu = 1$. We note that we can now define a notion of temperature T that is invariant for all observers. The scalar $\vec{u}^2$ is the modulus of the spatial fluid velocity. Generically the fluid velocity can be decomposed as $u^\mu = -v^\mu + \vec{u}^\mu$, where $\vec{u}^\mu = \hat{h}^\mu_\nu u^\nu$ with $\hat{h}^\mu_\nu \coloneqq \hat{h}^{\mu\rho}h_{\rho\nu}$. We furthermore define $\vec{u}_\mu = h_{\mu\nu}u^\nu = h_{\mu\nu}\vec{u}^\nu$, such that $\vec{u}^\mu = \hat{h}^\mu_\nu u^\nu$, but $\vec{u}^\mu \neq h^{\mu\nu}\vec{u}_\nu$. Note in particular that u^μ decomposes as follows relative to the Carrollian structure:

$$u^\mu = -v^\mu(1 - \theta^\nu\vec{u}_\nu) + h^\mu_\nu u^\nu. \quad (15)$$

The hydrostatic partition function at ideal order is given by $S = \int d^{d+1}x\, eP(T, \vec{u}^2)$. Using the general action variation (8) together with the "variational calculi" $\delta h^{\mu\nu} = 2v^{(\mu}h^{\nu)\rho}\delta\tau_\rho - h^{\mu\rho}h^{\nu\sigma}\delta h_{\rho\sigma}$ and $\delta v^\mu = v^\mu v^\nu\delta\tau_\nu - h^{\mu\nu}v^\rho\delta h_{\rho\nu}$ we obtain the ideal order currents:

$$\begin{aligned}
T^\mu_{(0)} &= Pv^\mu + sTu^\mu + m\vec{u}^2u^\mu,\\
\mathcal{T}^{\mu\nu}_{(0)} &= Ph^{\mu\nu} + mu^\mu u^\nu - 2(sT + m\vec{u}^2)u^{(\mu}\theta^{\nu)},\\
K_{(0)\mu} &= (sT + m\vec{u}^2)\vec{u}_\mu,
\end{aligned} \quad (16)$$

where the subscript (0) indicates that the currents are of ideal order $\mathcal{O}(1)$ and the entropy s and mass density m are defined via $dP = sdT + md\vec{u}^2$. The associated EMT is given by

$$T^{\mu}_{(0)\nu} = P\delta^{\mu}_{\nu} + mu^{\mu}\vec{u}_{\nu} - (sT + m\vec{u}^2)(u^{\mu}\hat{\tau}_{\nu} + \theta^{\mu}\vec{u}_{\nu}), \quad (17)$$

which transforms as in (10).

The equation of motion for the Goldstone $K_{(0)\mu} = 0$, which is equivalent to the boost Ward identity, gives a constraint on the dynamics

$$(sT + m\vec{u}^2)\vec{u}_{\mu} = 0, \quad (18)$$

and can be viewed as a framid condition for Carrollian fluids. Defining the energy density as $\mathcal{E} = \hat{\tau}_{\mu}T^{\mu}_{(0)}$, the Goldstone equation has two branches of solutions: either $\mathcal{E} + P = sT + m\vec{u}^2 = 0$ or $\vec{u}_{\mu} = 0$. Neither of them allows for the elimination of the Goldstone θ^{μ} from the low-energy description. The constraint (18) was derived in equilibrium, but we show in Appendix B of [46] that it also holds off equilibrium, although it receives corrections due to dissipative effects. As such, together with (11), it provides the ideal order dynamics for Carrollian fluids. Equations (16)–(18) are a central result of this work as they provide a well-defined notion of Carrollian fluids. Below we show that the $c \to 0$ limit of relativistic fluids gives rise to a Carrollian fluid with $\vec{u}_{\mu} = 0$.

The $c \to 0$ limit of a relativistic fluid.—The $c \to 0$ limit of relativistic fluids was considered in [17,18,20] (see also [23]) for a specific class of metrics. The same limit was taken in [24], where it was referred to as a "timelike fluid." Here, we demonstrate that these notions coincide and correspond to the special case of the Carrollian fluid we introduced above with $\vec{u}_{\mu} = 0$, and that the emergence of the Goldstone can be understood from the ultralocal expansion of the Lorentzian geometry. The relativistic EMT is given by

$$T^{\mu}{}_{\nu} = \frac{\hat{\mathcal{E}} + \hat{P}}{c^2}U^{\mu}U_{\nu} + \hat{P}\delta^{\mu}_{\nu}, \quad (19)$$

where $U^{\mu}U^{\nu}g_{\mu\nu} = -c^2$, and where the "hat" indicates relativistic thermodynamic quantities. To take the limit, we first consider the metric and its inverse in "pre-ultra-local (PUL) variables" [3]

$$g_{\mu\nu} = -c^2 T_{\mu}T_{\nu} + \Pi_{\mu\nu}, \quad g^{\mu\nu} = -\frac{1}{c^2}V^{\mu}V^{\nu} + \Pi^{\mu\nu}, \quad (20)$$

where $T_{\mu}V^{\mu} = -1$, $T_{\mu}\Pi^{\mu\nu} = V^{\mu}\Pi_{\mu\nu} = 0$, $\Pi^{\mu\rho}\Pi_{\rho\nu} = \delta^{\mu}_{\nu} + V^{\mu}T_{\nu}$. The leading order components of the PUL variables correspond to the fields that make up the Carrollian structure, e.g., $V^{\mu} = v^{\mu} + \mathcal{O}(c^2)$. We write the expansion of the relativistic fluid velocity relative to the PUL variables as

$$U^{\mu} = -V^{\mu} - c^2\mathbf{u}^{\mu},$$

for some $\mathbf{u}^{\mu}$. Crucially, U^{μ} is invariant under local Lorentz boosts while $\delta_C V^{\mu} = c^2 h^{\mu\nu}\lambda_{\nu} + \mathcal{O}(c^4)$, implying that $\delta_C \mathbf{u}^{\mu} = -h^{\mu\nu}\lambda_{\nu} + \mathcal{O}(c^2)$. This shows that $\mathbf{u}^{\mu}$ cannot be identified with a fluid velocity in the Carrollian limit. Indeed, we may identify the spatial part of the leading order term in the c^2 expansion of $\mathbf{u}^{\mu}$ with the spatial part of the boost Goldstone

$$\Pi_{\mu\nu}\mathbf{u}^{\nu} = h_{\mu\nu}\theta^{\nu} + \mathcal{O}(c^2) =: \vec{\theta}_{\mu} + \mathcal{O}(c^2). \quad (21)$$

Using this, together with $U_{\mu} = c^2\hat{\tau}_{\mu} + \mathcal{O}(c^4)$ the EMT becomes

$$T^{\mu}_{\nu} = (\mathcal{E} + P)v^{\mu}\hat{\tau}_{\nu} + P\delta^{\mu}_{\nu} + \mathcal{O}(c^2), \quad (22)$$

where $\mathcal{E}$ and P are the leading order contributions of $\hat{\mathcal{E}}$ and $\hat{P}$, respectively, satisfying the Euler relation $\mathcal{E} + P = sT$. This is exactly the "timelike" fluid of [24], corresponding to the $\vec{u}_{\mu} = 0$ branch of the Carrollian fluid we described above [70].

It is instructive to take the $c \to 0$ limit of the relativistic equation of motion $\hat{\nabla}_{\mu}T^{\mu}_{\nu} = 0$, where $\hat{\nabla}$ is the Levi-Civita connection of the spacetime metric $g_{\mu\nu}$. Deferring the details to Appendix C of [46], we note here that the equations of motion in the limit $c \to 0$ can be expressed as

$$\begin{aligned} v^{\mu}\partial_{\mu}\mathcal{E} &= (\mathcal{E} + P)K, \\ h^{\mu\nu}\partial_{\nu}P &= -\tilde{\varphi}^{\mu}(\mathcal{E} + P) + (\mathcal{E} + P)Kh^{\mu\nu}\vec{\theta}_{\nu} \\ &\quad - h^{\mu\nu}v^{\rho}\tilde{\nabla}_{\rho}[(\mathcal{E} + P)\vec{\theta}_{\nu}], \end{aligned} \quad (23)$$

where $K = h^{\mu\nu}K_{\mu\nu} := -\frac{1}{2}h^{\mu\nu}\pounds_{v}h_{\mu\nu}$ is the trace of the intrinsic torsion of the Carrollian structure [71], and is sometimes referred to as the "Carrollian expansion," while $\tilde{\varphi}^{\lambda} = 2h^{\lambda\mu}v^{\nu}\partial_{[\nu}\tau_{\mu]} - h^{\lambda\mu}h^{\nu\sigma}\vec{\theta}_{\sigma}K_{\mu\nu}$. The first term in $\tilde{\varphi}^{\lambda}$ is sometimes referred to as the "Carrollian acceleration," while the second term comes from the c^2 expansion of the Levi-Civita connection. The covariant derivative $\tilde{\nabla}$ is a Carroll compatible connection that arises in the $\mathcal{O}(1)$ piece of the c^2 expansion of the Levi-Civita connection, which we discuss further in Appendix C of [46]. These equations are fully covariant and reduce to the special case of the equations of motion obtained in [17,18,20,23] when restricted to spacetime metrics that admit a Randers-Papapetrou parametrization. Furthermore, these equations can be obtained by projecting the conservation law (11) along the time and spatial directions using (17) with $\vec{u}_{\mu} = 0$. We thus have shown that the "timelike" fluid of [24] is the same as the Carrollian fluid of [17,18,23], and both are a special case of the Carrollian fluid derived here.

Dissipation and modes.—In Appendix B of [46] we show that at order $\mathcal{O}(\partial)$ the class of Carrollian fluids we introduced is characterized by two hydrostatic coefficients and ten dissipative coefficients. Here we study the effect of specific coefficients in the linear spectrum of fluctuations. We consider flat Carrollian space with $\tau_{\mu} = \delta^{t}_{\mu}$, $v^{\mu} = -\delta^{\mu}_{t}$,

$h_{\mu\nu} = \delta^i_\mu \delta^i_\nu$, and $h^{\mu\nu} = \delta^\mu_i \delta^\nu_i$ (see Appendix B of [46] for more details). We then fluctuate the conservation equations (11) and the boost Ward identity (9) around an equilibrium state with constant temperature T_0, fluid velocity v^i_0, and Goldstone field θ^i_0, such that, e.g., $\theta^i = \theta^i_0 + \delta\theta^i$. Using plane wave perturbations with frequency ω and wave vector $\vec{k}$ we find a distinguishing feature of these Carrollian fluids: the mode structure strongly depends on whether the equilibrium state carries nonzero velocity v^i_0. If $v^i_0 = 0$, the linearized equations only admit a nontrivial solution if $\theta^i_0 \neq 0$, $T_0 = 0$, and $\delta v^i = 0$. Denoting the angle between the momentum k_i and θ^i_0 by ϕ, this leads to a single linear mode:

$$\omega = -\frac{1}{|\vec{\theta}_0| \cos\phi} |\vec{k}|, \tag{24}$$

where we assumed that the value of the entropy density s in equilibrium s_0 remains finite and nonvanishing when $T_0 \to 0$; otherwise there is no mode. Interestingly, this mode is not affected by any of the 12 transport coefficients entering at order $\mathcal{O}(\partial)$ [72]. This spectrum corresponds to the branch of solutions with $\vec{u}_\mu = 0$, and hence it is the expected spectrum arising from the $c \to 0$ limit of an ideal relativistic fluid.

On the other hand if $v^i_0 \neq 0$ but $\theta^i_0 = 0$ a more interesting spectrum can be obtained. For simplicity we only consider the effect of a bulk viscosity s_3 and one anisotropic viscosity s_2. Besides a gapped mode, we find for $d = 2$ a single diffusive mode of the form

$$\omega - v^i_0 k_i = -\frac{i\Gamma_1}{2} (\varepsilon_{ij} v^i_0 k^j)^2, \tag{25}$$

where ε_{ij} is the two-dimensional Levi-Civita symbol, and

$$\Gamma_1 = s_{2,0} \frac{[T_0^2 \chi_{TT} + |v_0|^2 (3T_0 \chi_{Tu} + 2|v_0|^2 \chi_{uu})]}{s_0 T_0 [T_0^2 \chi_{TT} + |v_0|^2 (2T_0 \chi_{Tu} + |v_0|^2 \chi_{uu})]} + \frac{2 s_{3,0}}{s_0 T_0},$$

where $s_{3,0}$ is the value of s_3 in equilibrium, ditto s_2, and where we defined $\chi_{TT} = (\partial^2 P/\partial T \partial T)_0$, $\chi_{uu} = (\partial^2 P/\partial \vec{u}^2 \partial \vec{u}^2)_0$, and $\chi_{Tu} = (\partial^2 P/\partial T \partial \vec{u}^2)_0$. The left-hand side of (25) is characteristic of a fluid without boost symmetry [63] while the right-hand side is typical of a diffusive mode. A salient signature of Carrollian fluids is that the spectrum is only nontrivial for states with nonzero equilibrium velocity v^i_0.

Discussion.—We have given a first-principles derivation of Carrollian fluids based on symmetries, showing that the spontaneous breaking of boost symmetry plays a crucial role in defining equilibrium partition functions of Carrollian field theories in the hydrodynamic regime. It is interesting to speculate whether this peculiar feature of Carrollian hydrodynamics can shed light on how to construct well-defined partition functions using specific microscopic models of Carrollian field theories along the lines of [24,44].

Different approaches to Carrollian hydrodynamics have appeared in the literature in the past few years [5,17–19,21–24]. Revisiting the $c \to 0$ limit of relativistic fluids we showed that there are subtleties regarding the interpretation of the dynamical variables that appear in the limit of the equations of motion. In particular, we showed that what naively appeared to be a fluid velocity was in fact a Goldstone field associated to the spontaneous breaking of boost symmetry. This allowed us to show that the different approaches are not only equivalent to each other but also special cases of the more general Carrollian fluids we introduced here. We believe it could be interesting to revisit the black hole membrane paradigm [2] in light of this new understanding.

The effective field theory geometry becomes Aristotelian once taking the Goldstone field into account. This allowed us to easily understand the dissipative structure of such fluids using earlier results [63,73]. The spectrum of excitations for certain classes of equilibrium states shares certain similarities with the spectrum of excitations of p-wave fracton superfluids in which the Goldstone field associated to the spontaneous breaking of dipole symmetry plays an analogous role to the boost Goldstone field [45,64,65], albeit in the Carroll case a nonvanishing fluid velocity is needed. We believe that this relation can be made clearer if we consider *strong* Carrollian geometries as we describe in Appendix D of [46].

Finally, it would be interesting to consider the addition of conformal symmetry as this could shine light on thermodynamic properties of holographic dual theories of flat space gravity. If we impose this symmetry the equation of state for the branch $\vec{u}_\mu = 0$ becomes $\mathcal{E} = dP$ while for the branch $sT + m\vec{u}^2 = 0$ it imposes the relation $m = -[(d+1)/\vec{u}^2]P$. In addition, the number of first order transport coefficients reduces from 12 to 8 (see Appendix B of [46]). It would be interesting to consider this case in further detail and explore its connections to flat space holography.

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[67] Alternatively, we could have introduced a Goldstone θ_a, where a is a tangent space index, but that would require us to work in a first order formalism.

[68] Note that we also have the boost invariant combination $\bar{h}_{\mu\nu} \coloneqq h^{\mu\nu} + 2v^{(\mu}\theta^{\nu)}$, but it does not form part of an Aristotelian structure, and it is not invariant under Stueckelberg transformations. The effective geometry obtained differs from that in [7] as the time component of the boost Goldstone is not physical and θ^{μ} is not a background field.

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